MICROCOPY   CHART

MRC Technical Summary Report #2897

ORBITAL COMPACTNESS AND ASYMPTOTIC
BEHAVIOR OF NONLINEAR PARABOLIC SYSTEMS
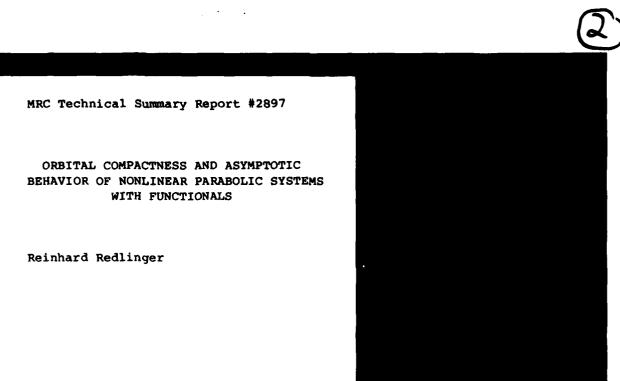WITH FUNCTIONALS

Reinhard Redlinger

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

January 1986

(Received November 22, 1985)

DTIC
ELECTE
MAY 2 3 1986
S D
D

86 5 20 156

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ORBITAL COMPACTNESS AND ASYMPTOTIC BEHAVIOR
OF NONLINEAR PARABOLIC SYSTEMS WITH FUNCTIONALS

Reinhard Redlinger

Technical Summary Report #2897

January 1986

ABSTRACT

Weakly coupled semilinear parabolic systems of the form $\dot{u} - A(x)u = g(u_t)$ with homogeneous boundary conditions are studied. The nonlinear function $g : C([-r,0] \times \Omega, R^n) \to R^n$ is assumed to be locally Lipschitz continuous with $r > 0$ a given real number and $\Omega \subset R^m$ a bounded domain, $\dot{u} = du/dt$, u_t for $t > 0$ is defined by $u_t(\sigma,\xi) = u(t+\sigma,\xi)$, $-r \leq \sigma \leq 0$, $\xi \in \Omega$ and A is a uniformly elliptic second order diagonal operator. Let u be a bounded classical solution. We first establish precompactness results for the orbit of u in several function spaces. Using these results and assuming that a Liapunov function V for the corresponding ordinary functional differential equation $\dot{z} = g(z_t)$ is known, we then show under some general conditions that the limit set ω^+ (as $t \to \infty$) of u consists of spatially homogeneous functions only. Moreover, ω^+ is invariant with respect to $\dot{z} = g(z_t)$ and $\dot{V} = 0$ on ω^+. The theory is illustrated with an example.

AMS (MOS) Subject Classifications: 35R10, 34G20

Key Words: Parabolic systems with functionals, orbital compactness, asymptotic behavior.

Work Unit Number 1 - Applied Analysis

SIGNIFICANCE AND EXPLANATION

In recent years, reaction-diffusion systems have become widely used as models in biology, chemistry and population dynamics.

A major point of interest is the long-time behavior of the solutions. For systems governed by ordinary differential equations the asymptotic behavior is usually investigated using Liapunov functionals in conjunction with an invariance principle. The purpose of this paper is to extend these methods to a general class of distributed systems that admit possible hysteresis effects in the reaction mechanism.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ORBITAL COMPACTNESS AND ASYMPTOTIC BEHAVIOR
OF NONLINEAR PARABOLIC SYSTEMS WITH FUNCTIONALS

Reinhard Redlinger

0. Introduction

Let $J = [-r,0]$ with $r > 0$ and Ω a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. We will consider in this paper weakly coupled nonlinear parabolic systems of the form $(k = 1,2,\ldots,n)$

$$\dot{u}^k - A^k(x)u^k = g^k(u_t) \quad \text{in} \quad D = [0,\infty) \times \Omega \ ,$$

$$(0.1) \qquad\qquad B^k(x)u^k = 0 \qquad \text{on} \quad [0,\infty) \times \partial\Omega \ ,$$

$$u^k = \varphi^k \qquad \text{in} \quad J \times \overline{\Omega} \ ,$$

where $u = (u^1,\ldots,u^n)$, $\dot{u}^k = \partial u^k/\partial t$, $g : C = C(J \times \overline{\Omega}, \mathbb{R}^n) \to \mathbb{R}^n$ is a given function, $\varphi \in C$, the A^k are uniformly elliptic operators of second order and the B^k linear boundary operators. As usually, $u_t \in C$ for $t > 0$ is defined by $u_t(\sigma,\xi) = u(t+\sigma,\xi)$, $\sigma \in J$, $\xi \in \overline{\Omega}$. We say that u is a classical solution of (0.1), if the function u together with its derivatives appearing in (0.1) is continuous in $\overline{D}$ such that equations (0.1) are identically satisfied. A bounded classical solution u is a classical solution with $\sup\{|u(t,x)| : -r < t, \ x \in \overline{\Omega}\} < \infty$.

In the first part of the present paper we will prove compactness results for the orbit $\Gamma(u) = \{u_t : t > 0\} < C$ of a bounded classical solution u of (0.1) in various function spaces. We give, in particular, sufficient conditions on A, B, g and φ under which $\Gamma(u)$ is relatively compact in the space

$$Y = \{v \in C : D_i v \in C, \ D_i D_j v(0,\cdot) \in C(\overline{\Omega},\mathbb{R}^n) \text{ for all } i, j\} \ ,$$

endowed with the norm

$$\|v\|_{(0,0)} = \|v\|_\infty + \sum_{i=1}^{m} \|D_i v\|_\infty + \sum_{i,j=1}^{m} \|D_i D_j v(0,\cdot)\|_\infty$$

$| \cdot |_\infty$ denoting the supremum norm and D_i the partial derivative $\partial/\partial x_i$. We deduce these results from more general theorems valid for abstract nonlinear evolution equations of the form

$$\dot{u} + Au = g(t, u_t) \quad , \quad t > 0 \quad ,$$

(0.2)

$$u_0 = \varphi \quad ,$$

where A is the infinitesimal generator of a strongly continuous analytic semigroup of linear operators in a Banach space X.

With (0.1) one can associate the system of ordinary functional differential equations

$$\dot{z} = \tilde{g}(z_t) \quad , \quad t > 0 \quad ,$$

(0.3)

$$z = \psi \quad \text{in } J \quad ,$$

with $\tilde{g}$ the restriction of g to the subspace C_h of spatially homogeneous functions in C. Assuming that a Liapunov function V for (0.3) is known and using the compactness results of the first part we show in section 3 that under some general conditions on the system (0.1) the limit set

$$\omega(u) = \{v \in C : \text{ There exists } t_k \to \infty \text{ with } u_{t_k} \to v \text{ in } C\}$$

of a bounded classical solution u of (0.1) consists of spatially homogeneous functions only. Moreover, $\omega(u)$ is contained in the largest invariant subset (with respect to the system (0.3)) of the set

$$S = \{\psi \in C_h : \dot{V}(\psi) = 0\} \quad .$$

In other words, the asymptotic behavior of solutions to (0.1) is completely determined by the behavior of the solutions z of (0.3). For systems (0.1) without functionals related results have been proven in [9] (see also [8] and the literature cited in these papers). The necessary compactness result in this case was established in [10]. We conclude the paper by treating in detail the example (n=1)

$$\dot{u} = \Delta u - \int_{t-r}^{t} a(s-t) |u(s,x)|^{\ell-1} u(s,x) ds \quad \text{in } (0,\infty) \times \Omega \quad ,$$

(0.4)

$$\partial u/\partial N = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega \quad ,$$

where N is the outer normal and $a \in C^2(J)$ is a given nonnegative, convex function with $a(-r) = 0$. We show that our results are applicable to (0.4) for all $\ell > 1$ in case $m = 1,2$ and for $1 < \ell < (m+2)/(m-2)$ otherwise.

-2-

1. The abstract equation.

Let X be a Banach space with norm $\|\cdot\|$ and let $A : D(A) \subset X$ be a closed, linear operator in X with domain of definition $D(A)$ dense in X. Throughout this section we will assume that

(1) $$\|(A + \lambda)^{-1}\| \le M(|\lambda| + 1)^{-1} \quad \text{for all} \quad \text{Re } \lambda > 0$$

with some constant $M > 0$ independent of λ (the norm in $L(X)$, the space of bounded linear operators from X to X, is also denoted by $\|\cdot\|$). It follows from (1) that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-At} : t \ge 0\}$ in X and that there are constants $C, \delta > 0$ such that (I denotes the identity operator)

(2) $$\|e^{-At}\| \le C e^{-\delta t} \quad , \quad \|A e^{-At}\| \le C t^{-1} e^{-\delta t} \quad \text{for} \quad t > 0 \quad ,$$

(3) $$\|(e^{-Ah} - I)A^{-1}\| \le Ch \quad \text{for} \quad h > 0$$

(cf. [13, Sect. 1]). This permits us to define the fractional power $A^{-\alpha}$ of A for any $\alpha > 0$ by the integral

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-As} s^{\alpha-1} ds \quad ,$$

where Γ denotes the gamma function. The operators $A^{-\alpha}$ are one-to-one and elements of $L(X)$. Hence it is reasonable to define $A^\alpha = (A^{-\alpha})^{-1}$. A^α *is a closed, densely defined linear operator in* X. With the norm $\|x\|_\alpha = \|A^\alpha x\|$, $X_\alpha = D(A^\alpha)$ becomes a Banach space. Set $A^0 = I$. Then, for any $\alpha < \beta < \gamma$, an inequality of moments

(4) $$\|x\|_\beta \le C_{\alpha\beta\gamma} \|x\|_\gamma^\lambda \|x\|_\alpha^{1-\lambda} \quad , \quad x \in X_\gamma$$

with $\lambda = (\beta-\alpha)/(\gamma-\alpha)$ is valid. The constant $C_{\alpha\beta\gamma}$ is independent of x. For proofs, see [13, l.c.].

Let $r > 0$ a real number and set $J = [-r, 0]$. Denote by $Z = C(J, X)$ the space of all continuous functions froms J to X with norm

$$\|\psi\|_Z = \sup\{\|\psi(s)\| : s \in J\} < \infty \quad .$$

If $b > 0$ and $u \in C([-r,b], X)$, then for $0 \le t \le b$, $u_t \in Z$ is defined by $u_t(s) = u(t+s)$, $s \in J$. Let $y : [0,\infty) \times Z \to X$ be a given function. In this section we will study continuity properties and boundedness in the spaces X_α, $0 \le \alpha < 1$, of solutions u of the initial value problem

$$\dot{u} + Au = g(t, u_t) \quad , \quad t > 0 ,$$

(5)

$$u_0 = \varphi \in Z ,$$

where, $\dot{u} = du/dt$. A strong solution u of (5) is a function $u \in C(J \cup R^+, X)$, whose restriction to $R^+ = [0, \infty)$ lies in $C(R^+, D(A)) \cap C^1(R^+, X)$, such that (5) is identically satisfied. A mild solution of (5) is a function $u \in C(J \cup R^+, X)$ with $u_0 = \varphi$ satisfying the integral equation

(6) $$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)}g(s, u_s)ds \quad , \quad t > 0 .$$

If $\sup\{|u(t)| : t > -r\} < \infty$, the solution is said to be bounded.

Let u be a fixed bounded mild solution of (5) and set $f(t) = g(t, u_t)$ for $t > 0$. Assume that there are constants $K, P > 0$ such that

(7a) $$|f(t)| < P \qquad \text{for } t > 0 ,$$

(7b) $$|f(t) - f(s)| < K(|t-s| + |u_t - u_s|) \quad \text{for } t, s > 0 .$$

For example, (7) holds if for any bounded set $B \subset Z$, the function g is (globally) Lipschitz continuous and bounded on $[0, \infty) \times B$.

Proposition 1. Let u be a bounded mild solution of (5), for which (7) is satisfied. Then, to any $0 < \alpha < 1$ with $\varphi(0) \in X_\alpha$, there is a constant $C_1 = C_1(\alpha)$ such that

$$|u(t)|_\alpha < C_1 \quad \text{for all } t > 0 .$$

Proof. Using (4), by (2), (7a) we get

$$|u(t)|_\alpha < C|\varphi(0)|_\alpha + \int_0^t C_{0\alpha 1}C(t-s)^{-\alpha}e^{-\delta(t-s)}P \, ds < \text{const.}$$

with a constant independent of t.

Proposition 2. Let u be a bounded strong solution of (5) with Lipschitz continuous initial value $\varphi \in C(J, X)$ and $\varphi(0) \in D(A)$. Assume (7). Then

$$|u(t+h) - u(t)| < C_2 h \quad \text{for all } t > 0, \, 0 < h < 1 ,$$

with a constant C_2 independent of t, h.

Before proving proposition 2 we first state an elementary lemma. For convenience, the following notation is introduced: Let $a, b, \tau : (0, \infty) \to R$ with $b(s) > 0$, $0 < \tau(s) < s$

for $s > 0$ and $b \in L^1_{loc}[0,\infty)$. For $y \in C[-r,\infty)$ define $Sy : (0,\infty) \to R$ by

$$(Sy)(t) = a(t) + \int_{\tau(t)}^{t} b(t-s) |y_s| ds \quad , \quad t > 0 \quad ,$$

where $|y_s| = \sup\{y(s+\sigma) : \sigma \in J\}$.

Lemma 3. Let S be defined as above and let $y,z \in C[-r,\infty)$ satisfy

$$y(t) - (Sy)(t) < z(t) - (Sz)(t) \quad \text{for} \quad t > 0 \quad ,$$

$$y(t) < z(t) \quad \text{for} \quad t \in J \quad ,$$

Then $y(t) < z(t)$ for all $t > -r$.

Proof. Since y, z are continuous and b is locally integrable, the functions Sy and Sz are well defined for $t > 0$. Suppose the assertion is false and let $\sigma = \inf\{t > -r : y(t) = z(t)\}$. Thus $y(\sigma) = z(\sigma)$, which implies $\sigma > 0$, and $y(t) < z(t)$ for $-r < t < \sigma$. But then at $t = \sigma$,

$$z = y - Sy + Sy < z - Sz + Sy < z - Sz + Sz = z \quad ,$$

a contradiction.

Proof of proposition 2. For any $0 < \tau < t$, $h > 0$ we have

(8) $$u(t+h) - u(t) = I_1 + I_2 + I_3 + I_4 \quad ,$$

where

$$I_1 = (e^{-Ah} - I)e^{-At} \varphi(0) \quad ,$$

$$I_2 = \int_{\tau}^{\tau+h} e^{-A(t+h-s)} f(s) ds \quad ,$$

$$I_3 = \int_{0}^{\tau} (e^{-Ah} - I)e^{-A(t-s)} f(s) ds \quad ,$$

$$I_4 = \int_{\tau}^{t} e^{-A(t-s)} (f(s+h) - f(s)) ds \quad .$$

By (2), (3) and (7a) we have

$$\|I_1\| < \|(e^{-Ah} - I)A^{-1}\| \, \|e^{-At}\| \, \|\varphi(0)\|_1 < C^2 h \|\varphi(0)\|_1 \quad ,$$

$$\|I_2\| < \int_{\tau}^{\tau+h} C \, P \, e^{-\delta(t+h-s)} ds < C \, P \, h$$

and (assuming $\tau < t$)

$$|I_3| < C P h \int_0^\tau |Ae^{-A(t-s)}| ds$$

$$< C^2 P h \int_0^\tau (t-s)^{-1} e^{-\delta(t-s)} ds < C^2 P(t-\tau)^{-1} \delta^{-1} h .$$

Choosing $\tau = \tau(t) = \max(0, t - (2CK)^{-1})$, we see that the function

$$y(t) = \|u(t+h) - u(t)\| \quad , \quad t > -r ,$$

satisfies the inequality

$$y(t) < C_3^! h + \int_\tau^t Ce^{-\delta(t-s)} K(h + |y_s|) ds$$

$$< C_3 h + \int_\tau^t CKe^{-\delta(t-s)} |y_s| ds \quad \text{for} \quad t > 0 ,$$

where $C_3^!$, C_3 are constants and $|y_s| = \sup\{y(s+\sigma) : \sigma \epsilon J\}$.

Our hypotheses imply that there is a constant $C_4 > 0$, independent of $h < 1$, with

$$y(t) < C_4 h \quad \text{for} \quad t \epsilon J .$$

Let

$$z(t) = Mh \quad \text{for} \quad t > -r ,$$

where $M > C_4$ is a constant. The assertion will follow from lemma 3, provided we can choose M such that

$$Mh > C_3 h + \int_\tau^t CKe^{-\delta(t-s)} Mh \, ds \quad \text{for all} \quad t > 0 .$$

But this is equivalent to

$$M > C_3 + CKM \delta^{-1}(1 - e^{-\delta(t-\tau)}) \quad \text{for} \quad t > 0 .$$

Hence it suffices to choose $M > \max(C_4, 2C_3)$.

Proposition 4. Under the hypotheses of proposition 2 there is to any $0 < \alpha < 1$ a constant $C_5 = C_5(\alpha)$ such that

$$\|u(t+h) - u(t)\|_\alpha < C_5 h^{1-\alpha} \quad \text{for all} \quad t > 0, \ 0 < h < 1 .$$

-6-

Proof. Choose $\tau = 0$ in (8). Then $I_3 = 0$ and

$$\|I_4\|_\alpha \leq \int_0^t \text{const.}(t-s)^{-\alpha} e^{-\delta(t-s)} h \, ds \leq \text{const. } h$$

by proposition 2. Further, using (4) we get

$$\|I_1\|_\alpha, \ \|I_2\|_\alpha \leq \text{const. } h^{1-\alpha} \ .$$

Since the constants are independent of $t > 0$, $0 < h \leq 1$, this gives the desired result.

Remark. Let $t_0 > 0$. Then the same method of proof shows $(0 < \alpha < 1)$

$$\|u(t+h) - u(t)\|_\alpha \leq \text{const. } h \quad \text{for} \quad t \geq t_0 > 0, \ 0 < h \leq 1 \ .$$

If u is only supposed to be a bounded mild solution of (5), then instead of proposition 4 we have

Proposition 5. Let u be a bounded mild solution of (5) such that (7) is satisfied. Assume $0 < \alpha < \rho < 1$ and $\varphi(0) \in X_\alpha$. Then

$$\|u(t+h) - u(t)\|_\alpha \leq c_6 h^{1-\rho}$$

for all $t, h > 0$ with a constant c_6 independent of t, h.

Proof. In view of proposition 1 we may assume $\rho < 1$. Choose $\tau = t$ in (8). Since $A^{-\varepsilon} \in L(X)$ for any $\varepsilon > 0$, we get

$$\|I_1\|_\alpha, \ \|I_2\|_\alpha \leq \text{const. } h^{1-\rho} \ .$$

Also

$$\|I_3\|_\alpha \leq \int_0^t \|A^\rho (e^{-Ah} - I)A^{-1}\| \ \|A^{\alpha+1-s} e^{-A(t-s)}\| \, P \, ds$$

$$\leq \text{const. } h^{1-\rho} \int_0^t (t-s)^{\rho-\alpha-1} e^{-\delta(t-s)} ds \leq \text{const. } h^{1-s} \ .$$

This proves the assertion.

Proposition 6. Let the hypotheses of proposition 2 be satisfied and assume that A^{-1} is compact. Then to any $0 < \alpha < 1$ and any $t_0 > 0$ there is a constant $C_7 = C_7(\alpha, t_0)$ with

$$\|\dot{u}(t)\|_\alpha \leq C_7 \quad \text{for all} \quad t \geq t_0 > 0 \ .$$

Proof. Let $\gamma = (1+\alpha)/2$. By the remark following proposition 4 we have

$$\|u(t+h) - u(t)\|_\gamma \leq \text{const. } h \quad \text{for all} \quad t \geq t_0, \ 0 < h \leq 1$$

with a constant independent of t, h. Since $A^{-\epsilon}$ for any $\epsilon > 0$ is a compact operator [4, p. 27] the assertion follows.

For the next proposition we assume that g is globally Lipschitz continuous in its second argument, i.e. that there is a constant $L > 0$ such that

(9) $\qquad \|g(t,\psi_1) - g(t,\psi_2)\| \leq L\|\psi_1 - \psi_2\|_Z$ for all $t > 0$, $\psi_1, \psi_2 \in Z$.

Proposition 7. Let u, $\hat{u}$ denote two mild solutions of (5) with initial values φ, $\hat{\varphi} \in Z$ resp. Assume (9). Then for all $t > 0$

(i) $\qquad\qquad \|u_t - \hat{u}_t\|_Z \leq C\|\varphi - \hat{\varphi}\|_Z e^{(LC-\delta)t}$ in case $CL - \delta > 0$,

(ii) $\qquad\qquad \|u_t - \hat{u}_t\|_Z \leq Ce^{\omega r}\|\varphi - \hat{\varphi}\|_Z e^{-\omega t}$ in case $CL - \delta < 0$,

where ω is the unique positive solution of $\omega + CL\, e^{\omega r} = \delta$.

Proof. Define $y(t) = \|u(t) - \hat{u}(t)\|$, $t > -r$, with $|y_t| = \sup\{y(t+\sigma) : \sigma \in J\}$ for $t > 0$. Then

$$y(t) \leq \|\varphi - \hat{\varphi}\|_Z \quad \text{for} \quad t \in J$$

and

$$y(t) \leq Ce^{-\delta t}|y_0| + \int_0^t CLe^{-\delta(t-s)}|y_s|ds \quad , \quad t > 0 \quad .$$

Let $t_+ = \max(0,t)$. Lemma 3 implies that for any $\epsilon > 0$ we have

(i) $\qquad\qquad y(t) < (C|y_0| + \epsilon)e^{(CL-\delta)t_+} \quad , \quad t > -r$

or resp.

(ii) $\qquad\qquad y(t) < (C+\epsilon)|y_0|e^{-\omega t_+} \quad , \quad t > -r \quad .$

Letting $\epsilon \to 0+$ the assertion follows in both cases.

Remark. In the foregoing proof we only made use of the bound $\|e^{-At}\| \leq Ce^{-\delta t}$ for $t > 0$ with δ not necessarily positive. Hence proposition 7 sharpens corollary 2.3 in [16] (where it is assumed that $C = 1$) and proposition 3.2 in [17] (in case $\alpha = 0$). Note that the assertions of corollaries 3.7 and 3.8 in [16] immediately follow from proposition 7. In contrast to [16, Section 3] we do not require g to be autonomous.

<u>Corollary</u>. To any $0 < \alpha < 1$, $t > 0$ with $\varphi(0) - \hat{\varphi}(0) \in X_\alpha$ there is a constant $C_8 = C_8(\alpha,t)$ such that

$$\| u(t) - \hat{u}(t) \|_\alpha < C \| \varphi(0) - \hat{\varphi}(0) \|_\alpha + C_8 \| \varphi - \hat{\varphi} \|_Z \ .$$

<u>Proof</u>. In case $CL - \delta > 0$ we have

$$\| u(t) - \hat{u}(t) \|_\alpha < \| e^{-tA}(\varphi(0) - \hat{\varphi}(0)) \|_\alpha$$

$$+ \int_0^t \| A^\alpha e^{-A(t-s)} \| \ L \ \| u_s - \hat{u}_s \|_Z \ ds$$

$$< C \| \varphi(0) - \hat{\varphi}(0) \|_\alpha + \int_0^t C_8'(t-s)^{-\alpha} e^{-\delta(t-s)} LC \| \varphi - \hat{\varphi} \|_Z \ e^{(LC-\delta)s} ds$$

$$< C \| \varphi(0) - \hat{\varphi}(0) \|_\alpha + \Gamma(1-\alpha)(LC)^\alpha C_8' e^{(LC-\delta)t} \| \varphi - \hat{\varphi} \|_Z \ .$$

The proof in case $CL - \delta < 0$ is similar.

2. Parabolic systems.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain whose boundary $\partial\Omega$ for some $0 < \mu < 1$ is a $(m-1)$-dimensional $C^{2+\mu}$-manifold such that Ω lies locally on one side of $\partial\Omega$, cf. [5, §4.4]. Define $\Gamma_0 = J \times \overline{\Omega}$ and let $C = C(\Gamma_0, \mathbb{R}^n)$ with supremum norm. In this section we will apply the results of section 1 to the weakly coupled parabolic system

$$(10a) \qquad \overset{\bullet}{u}{}^k + L^k u^k = g^k(t, x, u_t) \quad \text{in} \quad D = [0, \infty) \times \Omega \ ,$$

$(k = 1, 2, \ldots, n)$, where $u = (u^1, \ldots, u^n) : \overline{D} \times \Gamma_0 \to \mathbb{R}^n$ is a vector valued function, $\overset{\bullet}{u}{}^k = \partial u^k / \partial t$ and $u_t \in C$ for $t \geqslant 0$ is defined by $u_t(\sigma, \xi) = u(t+\sigma, \xi), (\sigma, \xi) \in \Gamma_0$. The operators L^k are given by

$$L^k = - \sum_{i,j=1}^{m} a_{ij}^k(x) D_i D_j + \sum_{i=1}^{m} a_i^k(x) D_i$$

with $D_i = \partial/\partial x_i$. We assume that the coefficient functions a_{ij}^k, a_i^k are μ-Hölder continuous in $\overline{\Omega}$ and that $(k = 1, 2, \ldots, n)$

$$\sum_{i,j=1}^{m} a_{ij}^k(x) \xi_i \xi_j \geqslant \lambda \sum_{i=1}^{m} \xi_i^2 \quad \text{for all} \quad \xi \in \mathbb{R}^m, \ x \in \overline{\Omega}$$

with some positive constant λ.

The boundary and initial conditions for (10a) are

$$(10b) \qquad B^k u^k = 0 \quad \text{on} \quad [0, \infty) \times \partial\Omega \ ,$$

$$(10c) \qquad u = \varphi \quad \text{in} \quad \Gamma_0 \ .$$

Here

$$B^k u^k \equiv b^k(x) u^k + \delta^k \partial u^k / \partial \beta^{(k)} \quad \text{for} \quad k = 1, 2, \ldots, n$$

with $\delta^k = 0$ or 1 and $b^k \in C^{1+\mu}(\partial\Omega)$. Also $\beta^k = \beta^k(x) \in C^{1+\mu}(\partial\Omega, \mathbb{R}^m)$ is an outward pointing, nowhere tangent vector field on $\partial\Omega$. In case $\delta^k = 0$ we assume that $b^k \equiv 1$ on $\partial\Omega$ and that the compatibility condition of first order is satisfied, i.e. that $L^k \varphi^k(0, x) - g^k(0, x, \varphi) = 0$ on $\partial\Omega$.

Let $g : \mathbb{R}^+ \times \overline{\Omega} \times C \to \mathbb{R}^n$ in (10a) be a given function and $\varphi \in C$ with $\varphi(0) \in C^2(\overline{\Omega}, \mathbb{R}^n)$ in (10c). Note that $B^k \varphi^k(0) = 0$ on $\partial\Omega$ by (10b).

A classical solution u of (10) is a function $u \in C(\Gamma_0 \cup \overline{D}, \mathbb{R}^n)$ whose restriction to $\overline{D}$ lies in $C^{1,2}(\overline{D}, \mathbb{R}^n)$, i.e. whose components are continuously differentiable in $\overline{D}$,

-10-

twice with respect to x and once with respect to t, such that equations (10) are identically satisfied. A bounded classical solution u of (10) is a classical solution with $\sup\{|u(t,x)| : (t,x) \in \Gamma_0 \cup \overline{D}\} < \infty$.

Let $X = L^p(\Omega, R^n)$, where $1 < p < \infty$ is fixed but arbitrary. For sufficiently large $d > 0$, the operator $A : D(A) \to X$ defined by

$$Au = (L^1 u^1, \ldots, L^n u^n) + du$$

with

$$D(A) = \{v \in W^{2,p}(\Omega, R^n) : B^k v^k = 0 \text{ on } \partial\Omega \text{ for all } k\}$$

satisfies the assumptions on A in section 1, cf. [2, §I.19]. Let u be a fixed bounded classical solution of (10) such that:

(H_1)
For some $p_0 > m$, the function $f(t) = g(t, \cdot, u_t) : R^+ \to X = L^{p_0}(\Omega, R^n)$ satisfies (7) .

Then, by propositions 1 and 5 there are constants $C_1 = C_1(\alpha)$, $C_6 = C_6(\alpha, \rho)$ such that

$$\|u(t)\|_\alpha < C_1 \quad , \quad \|u(t) - u(s)\|_\alpha < C_6 |t-s|^{1-\rho}$$

for all $t, s > 0$, $0 < \alpha < \rho < 1$. But

(11)
$$X_\alpha \subset C^\lambda(\overline{\Omega}, R^n) \quad \text{for} \quad 0 < \lambda < 2\alpha - m/p$$

with continuous imbedding, cf. [4, Thm. 1.6.1]. Hence, defining $\nu_0 = 1 - m/p_0$ we see that for any $0 < \nu < \nu_0$

(12)
$$u(t, \cdot) \in C^{1+\nu}(\overline{\Omega}, R^n) \quad \text{for all} \quad t > 0$$

with uniformly bounded norm. Moreover, for any $x \in \overline{\Omega}$ we have $(i = 1, 2, \ldots, m)$

(13)
$$u(\cdot, x) \in C^\nu(R^+, R^n), \quad D_i u(\cdot, x) \in C^{\nu/2}(R^+, R^n) \quad ,$$

where again $0 < \nu < \nu_0$ is arbitrary, with Hölder norms bounded uniformly in x.

For $0 < \nu < 1$ define

$$Y_\nu = \{v \in C^{0,1}(\Gamma_0, R^n) : \|v\|_{(\nu)} < \infty\}$$

with

-11-

$$\|v\|_{(\nu)} = \|v\|_\infty + [v]_{\nu,x} + [v]_{\nu,t}$$
$$+ \sum_{i=1}^m (\|D_i v\|_\infty + [D_i v]_{\nu,x} + [D_i v]_{\nu/2,t}) \quad,$$

where, for $\psi \in C$ and $0 < \alpha < 1$,

$$\|\psi\|_\infty = \sup\{|\psi(\sigma,\xi)| : (\sigma,\xi) \in \Gamma_0\} \quad,$$

$$[\psi]_{\alpha,t} = \sup\{|\psi(t,\xi) - \psi(t,\eta)| \ |\xi-\eta|^{-\alpha} : t \in J, \ \xi \neq \eta \in \overline{\Omega}\} \quad,$$

$$[\psi]_{\alpha,t} = \sup\{|\psi(t,\xi) - \psi(s,\xi)| \ |t-s|^{-\alpha} : \xi \in \overline{\Omega}, \ t \neq s \in J\} \quad.$$

Since for $0 < \nu < \rho < 1$, Y_ρ is compactly embedded in Y_ν, it follows from (12), (13) that the orbit $\Gamma(u) = \{u_t : t > 0\}$ of a bounded classical solution u of (10) is relatively compact in Y_ν for any $0 < \nu < \nu_0$ provided $\varphi = u_0 \in Y_\rho$ for some $\rho > \nu$. Further, if φ is sufficiently smooth, say $\varphi \in Y_1$, then proposition 6 gives

(14) $$\dot{u}(t,\cdot) \in C^\nu(\overline{\Omega},\mathbb{R}^n) \quad \text{for all} \quad t > t_0 > 0$$

with arbitrary $0 < \nu < \nu_0$, $t_0 > 0$. The Hölder norms are bounded uniformly in x. Write (10a,b) as an uncoupled system of linear elliptic equations

$$L^k u^k = h^k(x;t) \quad \text{in} \quad \Omega \quad,$$
(15)
$$B^k u^k = 0 \quad \text{on} \quad \partial\Omega \quad,$$

where $h^k(x;t) = g^k(t,x,u_t) - \dot{u}^k$ depends on t as a parameter. Suppose that g has the following property:

(H_2) If $\psi \in Y_\nu$ for some $0 < \nu < 1$ then, for any fixed

   $t > 0$, the function $h(x) = g(t,x,\psi)$ is ρ-Hölder

   continuous in $x \in \overline{\Omega}$ with $\rho = \rho(\nu)$, the ρ-Hölder

   norm of h depending only on ν and $\|\psi\|_{(\nu)}$.

Then, by (14) and (H_2) the functions $h(x;t)$, $t > t_0$ are ω-Hölder continuous in x, uniformly with respect to t, for some $0 < \omega < 1$. Hence, by the Schaunder estimates $u(t,\cdot) \in C^{2+\tilde{\omega}}(\overline{\Omega},\mathbb{R}^n)$ for all $t > t_0 > 0$ with uniformly bounded norm, where $\tilde{\omega} = \min(\mu,\omega)$ (cf. [6, Chap. 3]).

   For $0 < \nu, \sigma < 1$ let

$$Y_{\nu,\sigma} = \{v \in Y_\nu : v(0,\cdot) \in C^2(\overline{\Omega},R^n) , \|v\|_{(\nu,\sigma)} < \infty\}$$

where

$$\|v\|_{(\nu,\sigma)} = \|v\|_{(\nu)} + \sum_{i,j=1}^{m} \|D_i D_j v\|_\infty + \sup\{|D_i D_j v(0,\xi) - D_i D_j v(0,n)| \, |\xi-n|^{-\sigma} : \xi \neq n \in \overline{\Omega}\} .$$

Then, $\Gamma_\tau(u) = \{u_t : t \geq \tau\}$ for any $\tau > 0$ is relatively compact in $Y_{\nu,\sigma}$ for all $0 < \nu < \nu_0$, $0 < \sigma < \widetilde{\omega}$. We summarize our results in two theorems.

Theorem 8. Let u be a bounded classical solution of (10) such that (H_1) is fulfilled. Assume that the initial value $\varphi \in C$ with $\varphi(0,\cdot) \in C^2(\overline{\Omega},R^n)$ satisfies $\varphi \in Y_\rho$ for some $0 < \rho < 1$. Then the orbit $\Gamma(u) = \{u_t : t > 0\}$ of u is relatively compact in Y_ν for any $0 < \nu < \min(\rho,\nu_0)$.

Theorem 9. Let u be a bounded classical solution of (10) satisfying (H_1), (H_2). Assume $\varphi \in Y_1$ with $\varphi(0,\cdot) \in C^2(\overline{\Omega},R^n)$. Then, $\Gamma_\tau(u) = \{u_t : t \geq \tau\}$ is relatively compact in $Y_{\nu,\sigma}$ for any $0 < \nu < \nu_0$, $0 < \sigma < \widetilde{\omega}$, where $\tau > 0$ is arbitrary.

Remarks: α) Under the hypotheses of theorem 8, $\Gamma_\tau(u)$ for $\tau > r$ is relatively compact in Y_ν for any $0 < \nu < \nu_0$. This is true even without the assumption $\varphi \in Y_\rho$. Similarly, if the assumption $\varphi \in Y_1$ in theorem 9 is replaced by $\varphi \in Lip(J, C(\overline{\Omega},R^n))$, then the assertion remains valid for $\tau > r$. Conversely, suppose that under the hypotheses of theorem 9 the function $k(t,x) = g(t,x,u_t)$ is an element of $C^{\widetilde{\omega}/2,\widetilde{\omega}}([0,t_0] \times \overline{\Omega},R^n)$ for some $t_0 > 0$ (see [5] for definition of this space). Then, by [5, Sect. 4.5] it follows that $\tau = 0$ can be admitted in theorem 9.

β) Let u be a bounded classical solution of (10) and assume: For any fixed $t > 0$ the function $g(t,\cdot,u_t) : \Omega \to R^n$ is measurable, and the set $\{g(t,x,u_t) : (t,x) \in \overline{D}\}$ is bounded. Then the function $f(t) = g(t,\cdot,u_t)$ satisfies (7a) in $X = L^p(\Omega,R^n)$ for any $p_0 > 1$ and the same is true for (7b) provided this inequality holds in $X = L^{p_0}(\Omega,R^n)$ for some $p_0 > 1$. Hence, in this case we can admit $\nu_0 = 1$ in the above considerations.

γ) Some simple examples for functionals that can be treated by the above method would be

$$g(t,x,\psi) = \psi(-r,x), \quad \psi(0,x)\psi(-r,x) \quad ,$$

$$\int_{|\xi|\leq|x|} \psi^2(0,\xi)d\xi \quad \text{and}$$

$$\int_{-r}^0 h(s)\psi(s,x)ds \quad \text{with} \quad h \in L^1(J) \quad .$$

On the other hand, there are functionals that do not fall within the scope of the L^p-theory developed above, as the example

(16) $$g(t,x,\psi) = \psi(0,x_0) \quad , \quad x_0 \in \Omega \quad \text{fixed}$$

shows. However, (16) can be admitted, if we choose $X = X_c = C(\overline{\Omega},R^n)$ with supremum norm and define the operator $A_c : D(A_c) \subset X_c \to X_c$ by $A_c v = Av, \ v \in D(A_c)$ with

$$D(A_c) = \{v \in X_c : v \in W^{2,q}(\Omega,R^n), \ Av \in X_c \quad ,$$

$$B^k v^k = 0 \quad \text{on} \quad \partial\Omega \quad \text{for all} \quad k\} \quad ,$$

where $q > m$ is a fixed, but arbitrary real number. By [14], [15] inequality (16) is then satisfied and from $X_c \subset X = L^p(\Omega,R^n)$ and $D(A_c) \subset D(A)$ for any $1 \leq p \leq q$ it follows that $D(A_c^\alpha) \subset D(A^\beta)$ for all $0 \leq \beta < \alpha \leq 1$. Hence, theorems 8 and 9 can be proved as above (with $\nu_0 = 1$ and, of course, X replaced by X_c in (H_1)), provided $D(A_c)$ is dense in X_c. But this is true only, if $\delta^k = 1$ in (10b) for all k. In general, we have to replace X_c by

$$\widetilde{X}_c = \{v \in X_c : v^k = 0 \quad \text{on} \quad \partial\Omega \quad , \quad \text{if} \quad \delta^k = 0 \quad \text{in} \quad (10b)\} \quad ,$$

which in turn implies an additional condition on the nonlinearity g in (10a), namely:

For any k with $\delta^k = 0$ in (10b) we have

$$g^k(t,x,u_t) = 0 \quad \text{for all} \quad t > 0, \ x \in \partial\Omega,$$

u denoting a bounded classical solution of (10).

In each concrete case, using (10b) this condition can be easily verified.

3. Asymptotic behavior.

Let $C_h = C(J, R^n)$ the subspace of spatially homogeneous functions in C. We will assume throughout this section that the nonlinearity g in (10a) is autonomous, i.e. that $g = g(u_t)$. Consider the ordinary functional differential equation

(17a)
$$\dot{z} = g(z_t) \quad \text{for} \quad t > 0 \ ,$$

(17b)
$$z_0 = \psi \in C_h \ .$$

For any set $G \subset R^n$ define the subspace C_G of C by

$$C_G = \{v \in C : v(\Gamma_0) \subset G\} \ .$$

Let $V : C_h \to R$ a continuous function and set

$$\dot{V}(\psi) = \overline{\lim_{h \to 0+}} \ h^{-1}(V(z_h) - V(\psi)) \quad \text{for} \quad \psi \in C_h \ ,$$

where $z = z(t;\psi)$ is the solution of (17a) with initial value $z_0 = \psi$. We say that $V : C_h \to R$ is a Liapunov function for (17) on $C_{G,h} = C_G \cap C_h$, if V is continuous on $\overline{C}_{G,h}$ and $\dot{V} \leq 0$ on $C_{G,h}$. Let

$$S = \{\psi \in \overline{C}_{G,h} : \dot{V}(\psi) \leq 0\}$$

and denote by

(18) M the largest set in S which is invariant

with respect to (17).

(A set $K \subset C_h$ is said to be invariant with respect to (17), if for any $\psi \in K$ there is a continuous curve $w : R \to K$ with $w(0) = \psi$ and $z_t(w(\tau)) = w(t+\tau)$ for all $t \geq 0$, $\tau \in R$.) We then have the following assertion [3, §13]:

If V is a Liapunov function on $C_{G,h}$ and if $z(t;\psi)$

is a bounded solution of (17) with values in G, then

$z_t \to M$ as $t \to \infty$.

Let V be a Liapunov function for (17) on $C_{G,h}$ and define $W : C \to R$ by

(19) $$W(\varphi) = \int_\Omega V(\varphi(\cdot,\xi))d\xi \ , \quad \varphi \in C_G \ .$$

Let $Y_{\nu,\sigma}$ with some $0 < \nu < 1$, $0 < \sigma < \tilde{\omega}$ be fixed in the sequel ($\tilde{\omega}$ is the constant appearing in theorem 9). We introduce the following hypotheses:

-15-

(H_3) For any $\varphi \in \overline{C}_G \cap Y_{\nu,\sigma}$ satisfying the compatibility
condition of first order the initial value problem (10)
has a unique classical solution u with $u_t \in \overline{C}_G$
for $t > 0$. Moreover, to any $T > 0$ there is a constant
P, depending only on T and $|\varphi|_{(\nu,\sigma)}$, such that
$|u(t,x)| < P$ for all $-r < t < T$, $x \in \overline{\Omega}$.

(H_4) There are functions $c^k \in C(\mathbb{R}^n, \mathbb{R}^+)$, which have only
isolated zeros, such that

$$\dot{W}(\varphi) = \overline{\lim_{h \to 0+}} \ h^{-1}(W(u_h) - W(\varphi))$$

$$< -\int_\Omega [\sum_{k=1}^n c^k(\varphi(0,\xi))|\nabla \varphi^k(0,\xi)|^2 + \dot{V}(\varphi(\cdot,\xi))]d\xi$$

for all $\varphi \in \overline{C}_G \cap Y_{\nu,\sigma}$, $u = u(\varphi)$ denoting the solution of
(10) with initial value φ.

Theorem 10. Let $\overline{u}$ be a bounded classical solution of (10), where $g = g(u_t)$ is
autonomous and locally Lipschitz continuous. Assume that the initial value $\overline{\varphi}$ of $\overline{u}$
satisfies $\overline{\varphi} \in Lip(J,C(\overline{\Omega},\mathbb{R}^n))$, $\overline{\varphi}(0,\cdot) \in C^2(\overline{\Omega},\mathbb{R}^n)$ and that (H_1), (H_2) are fulfilled for
$\overline{u}$, g. Let the values of $\overline{u}$ lie in a set $G \subset \mathbb{R}^n$ and let there exist a Liapunov
function V for system (17) on $C_{G,h}$. Assume (H_3) and (H_4).

Then $\overline{u}_t \to M$ as $t \to \infty$ in C, where M is defined by (18), i.e. the asymptotic
behavior of $\overline{u}$ is uniquely determined by the asymptotic behavior of the solutions z of
the ordinary functional differential equation (16).

Proof. Let

$$\omega(\overline{u}) = \{v \in C : \text{There exists } t_k \to \infty \text{ with } \overline{u}_{t_k} \to v \text{ in } C\} \ .$$

Then

$$\omega(\overline{u}) = \bigcap_{\tau > 0} \overline{\Gamma_\tau(\overline{u})}$$

-16-

and by theorem 9 $\omega(\overline{u})$ is nonempty and compact in $Y_{\nu,\sigma}$. Let $\varphi_0 \in \omega(\overline{u})$ be arbitrary. Then $\varphi_0 \in \overline{C}_G \cap Y_{\nu,\sigma}$ and the solution $u_0 = u_0(\varphi_0)$ of (10) with initial value φ_0 is well defined by (H$_3$) (note that φ_0 satisfies the compatibility conditions). Using a cut-off procedure we see by (H$_1$), (H$_3$) and the corollary to proposition 7 that for every $t > 0$ (assuming $\overline{u}_{t_k} \to \varphi_0$)

$$\overline{u}_{t+t_k} = u_t(\overline{u}_{t_k}) \to u_{0,t}(\varphi_0) \quad \text{in} \quad C \quad (k \to \infty) \quad .$$

Hence $\omega(\overline{u})$ is positively invariant.

By compactness, there exists a subsequence $t_{n_1} \to \infty$ such that $\lim_{n_1 \to \infty} u_{t_{n_1}} - 1 = \varphi_1$

exists in $\overline{C}_G \cap Y_{\nu,\sigma}$. Taking further subsequences and then the diagonal subsequence in the usual way we find $t_{n'} \to \infty$ such that

$$u_{t_{n'}-j} \to \varphi_j \quad \text{in} \quad \overline{C}_G \cap Y_{\nu,\sigma} \quad \text{as} \quad n' \to \infty \quad \text{for} \quad j = 0,1,2,\dots \quad .$$

Define a curve $w : R \to C$ by

$$w(t) = u_{t+j}(\varphi_j) \quad \text{for} \quad t > -j \quad (j = 0,1,2,\dots) \quad .$$

This is consistent because

$$u_{t+j}(\varphi_j) = u_{t+k}(\varphi_k) \quad \text{for} \quad t > -j > -k \quad .$$

It follows that $\omega(\overline{u})$ is an invariant set with respect to (10).

By hypothesis (H$_4$) the function $W(\overline{u}_t)$ is decreasing for, say, $t > r$. Since it is bounded below, W must be constant on $\omega(\overline{u})$. Hence $\dot{W}(\varphi_0) = 0$ for any $\varphi_0 \in \omega(\overline{u})$. This implies in particular that $\varphi_0(0,x)$ is independent of x by the assumptions on the c^k in (H$_4$). But then $\varphi_0(t,x)$ is independent of x for all $t \in J$ by the invariance of $\omega(\overline{u})$. We thus see that $\omega(\overline{u}) \subset M$, where M is defined by (18), and the proof is complete.

Remarks: (i) If g is globally Lipschitz continuous in C, then sufficient conditions for (H$_3$) to hold can be derived from the results in [11] using [5, Section 4.5]. In the general case of only locally Lipschitz continuous g the crucial point is the derivation of an a priori bound for $|u|$ (depending of course on $\|\varphi\|_{(\nu,\sigma)}$). To this end, the following two methods can be used:

α) comparison arguments, cf. [11];

β) functional analytic methods (feedback arguments), cf. section 4 and [12].

(ii) Our assumptions on the initial function φ are quite strict (compatibility conditions, smoothness, boundary behavior). It should be clear, however, that the results of this section are valid under any conditions on φ that will insure the solution to be classical for $t > t_0$ for some $t_0 > 0$. We then simply consider the initial boundary value problem (10) on the time interval $[t_0 + r, \infty)$.

(iii) The proof of theorem 10 follows [4, Section 4.3]. As in [4, l.c] one can additionally show that $\omega(\bar{u})$ is connected in C.

(iv) As shown by the example studied in [7], in special cases the ansatz (19) may also be useful, if g depends on x as well as on u_t.

4. An example

We consider the initial boundary value problem $(n=1)$

$$(20a) \qquad \dot{u} = \Delta u - \int_{-r}^{0} a(s) \ h(u(t+s,x)) ds \quad \text{in} \quad D \ ,$$

$$(20b) \qquad \partial u / \partial N = 0 \quad \text{on} \quad (0,\infty) \times \partial \Omega \ ,$$

$$(20c) \qquad u = \varphi \quad \text{in} \quad \Gamma_0 \ ,$$

where N denotes the outer normal to $\partial \Omega$ and

$$h(z) = |z|^{\ell-1} z \ , \quad z \in R, \quad \text{for some} \quad \ell > 1 \ .$$

The corresponding partial differential equation

$$\dot{v} = \Delta v - |v|^{\ell-1} v \quad \text{in} \quad D$$

has been used as a model equation by several authors, see e.g. [1].

Let us assume that the density a in (20a) satisfies:

$$(21) \qquad a \in C^2(J) \text{ is a nonnegative convex function, with } a(-r) = 0$$

(cf. [3, §14]). Following the procedure outlined in section 3 and using the results of [3]
we define a function $W : C \rightarrow R$ by

$$W(\varphi) = \int_{\Omega} V(\varphi(\cdot,\xi)) d\xi \ , \quad \varphi \in C \ ,$$

where

$$V(\psi) = H(\psi(0)) + \frac{1}{2} \int_{-r}^{0} \ddot{a}(s) [\int_{s}^{0} h(\psi(\sigma)) d\sigma]^2 ds \ , \quad \psi \in C_h$$

with

$$H(z) = \int_{0}^{z} h(z) dz \ , \quad \text{i.e.} \quad H(z) = |z|^{\ell+1}/(\ell+1) \ , \quad z \in R \ .$$

We get (for smooth φ)

$$\dot{W}(\varphi) = \int_{\Omega} h(\varphi(0,\xi)) \Delta\varphi(0,\xi) d\xi + \int_{\Omega} \dot{V}(\varphi(\cdot,\xi)) d\xi$$

$$= -\int_{\Omega} h'(\varphi(0,\xi)) |\nabla\varphi(0,\xi)|^2 d\xi + \int_{\Omega} \dot{V}(\varphi(\cdot,\xi)) d\xi$$

$$(22)$$

$$\dot{V}(\psi) = -\frac{1}{2} \dot{a}(-r) \ [\int_{-r}^{0} h(\psi(s)) ds]^2$$

$$- \frac{1}{2} \int_{-r}^{0} \ddot{a}(s) \ [\int_{s}^{0} h(\psi(\sigma)) d\sigma]^2 ds \ .$$

Hence W is a Liapunov function for (20) satisfying (H_4) with $G = R$. (Note that $\ddot{a} \geq 0$
in J by (21).)

Now assume that the initial function φ in (20c) is such that

(23)
$$\varphi \in C^{\varepsilon/2,\varepsilon}(\Gamma_0) \quad \text{with} \quad \varphi(0,\cdot) \in C^{2+\varepsilon}(\overline{\Omega}) \quad \text{for some} \quad \varepsilon > 0 \quad .$$

$$\text{Also} \quad \partial\varphi/\partial N = 0 \quad \text{on} \quad \partial\Omega \quad .$$

It then follows from [11, Thms. 3.1 and 3.3] that for some $T > 0$ the boundary value

problem (20) has a unique continuous solution $u \in C([-r,T] \times \overline{\Omega})$. Moreover, this solution

is classical for $t > 0$. We will show next that there is an a priori bound for $|u|$,

independent of T, provided that $m < 2(\ell+1)/(\ell-1)$.

In fact, it follows from (22) that $W(u_t) \leq W(u_\tau)$ for any $0 < \tau \leq t$ and hence by

continuity

$$W(u_t) \leq W(u_0) = W(\varphi) \quad \text{for} \quad 0 \leq t < T \quad ,$$

which gives an a priori bound for $u(t,\cdot)$ in $L^{\ell+1}(\Omega)$ independent of t.

To (20) corresponds the abstract integral equation

(24)
$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} k(u_s) ds$$

with $u_0 = \varphi$, where the operator A in $X = L^p(\Omega)$, $1 < p < \infty$, is defined as in section 2

and

$$k(u_t) = -\int_{-r}^0 a(s) h(u(t+s)) ds \quad , \quad t > 0 \quad .$$

For $0 < t < T$, u is a solution of (24). Let $0 < \alpha < 1$. Then by (2), (4)

$$\|u(t)\|_\alpha \leq C_{\alpha 1} t^{-\alpha} \|\varphi(0)\|_{\langle p \rangle} + C_{\alpha 2} \sup\{\|k(u_t)\|_{\langle p \rangle} : 0 < t < T\}$$

for $0 < t < T$ with constants $C_{\alpha i}$ independent of t and $\|\cdot\|_{\langle p \rangle}$ denoting the norm in

$L^p(\Omega)$. But

(25)
$$\|k(\psi)\|_{\langle p \rangle} \leq K \sup\{\|\psi(\sigma)\|^\ell_{\langle q \rangle} : \sigma \in J\} \quad \text{for} \quad \psi \in C$$

with $1 < p, q < \infty$, $q = \ell p$ and the constant K independent of ψ. Furthermore, since

u is continuous on $[-r,T] \times \overline{\Omega}$, there is a $t_0 > 0$ such that, say,

$$|u(t,x)| \leq 2(\|\varphi\|_\infty + 1) \quad \text{for} \quad -r \leq t \leq t_0, \ x \in \overline{\Omega} \quad .$$

(Using comparison functions of the form $M(1 - Kt)^{-1}$ with K, M constant, it is easy to

give an explicit positive lower bound for t_0 in terms of $\|\varphi\|_\infty$, cf. [11, Thm. 3.4].)

By [4, Thm. 1.6.1]

(26)
$$D(A^\alpha) \subset L^\rho(\Omega) \quad , \quad \text{if} \quad -\frac{m}{\rho} < 2\alpha - \frac{m}{p} \quad , \quad 1 < \rho < \infty \quad .$$

Hence, summing up and using (11), (26) we see that an a priori bound on $u(t)$ in $L^q(\Omega)$

implies an a priori bound on $u(t)$ in

$$C(\overline{\Omega}) \quad \text{in case} \quad 2q > m\ell$$

and in

$$L^\rho(\Omega) \quad \text{in case} \quad 2q < m\ell \quad , \quad 1 < \rho < (\frac{\ell}{q} - \frac{2}{m})^{-1} \quad ,$$

both bounds being independent of t.

We start with a bound in $L^{\ell+1}(\Omega)$. If $m < 2(1 + \ell^{-1})$ this leads to a bound in $C(\overline{\Omega})$ and we are done. If $m > 2(1 + \ell^{-1})$ we get a bound in $L^\rho(\Omega)$ with ρ being arbitrary close to $(\frac{\ell}{\ell+1} - \frac{2}{m})^{-1}$. Thus we can choose $\rho > \ell+1$ if and only if

(27) $$m < 2\frac{\ell+1}{\ell-1} \quad .$$

Assuming (27) and repeating the above reasoning we get a nondecreasing sequence (ρ_k) of real numbers with

$$\rho_1 = \rho < (\frac{\ell}{\ell+1} - \frac{2}{m})^{-1} \quad , \quad \rho_{k+1} < (\frac{\ell}{\rho_k} - \frac{2}{m})^{-1}$$

such that $u(t)$ is bounded in $L^{\rho_k}(\Omega)$, uniformly with respect to t. But it is easy to show that after finitely many steps ρ_k can be chosen so as to fulfill $2\rho_k > m\ell$. In the next step this leads to the desired a priori bound for $u(t)$ in $C(\overline{\Omega})$. Hence, using a cutting-off procedure we can transform (20a) into an equation with globally Lipschitz continuous nonlinearity, u still being a solution of the transformed equation on its interval of existence. But then it follows from [11, Thm. 3.1] that u is a global solution of (20). Moreover, u is bounded and classical for $t > 0$. We can thus apply the results of section 3 to this solution. In view of [3, §14] we arrive at the following

Proposition 11. Assume (21), (23) and (27). Then the boundary value problem (20) has a unique bounded classical solution u. As $t \to \infty$, this solution converges (uniformly with respect to $x \in \overline{\Omega}$) to 0 provided $\ddot{a}(s) \neq 0$ for some $s \in J$, and to a r-periodic solution of the second order equation $\ddot{z} + a(0)h(z) = 0$ in case a is linear, $a \not\equiv 0$.

Remarks. (i) If we replace the boundary condition (20b) by $u = 0$ on $(0,\infty) \times \partial\Omega$, then it follows by the same reasoning as above that for smooth initial values φ this boundary value problem has a unique bounded classical solution $\tilde{u}$. As $t \to \infty$ this solution converges to zero, uniformly with respect to $x \in \overline{\Omega}$. The admissible values of ℓ and m

are again given by (27).

(ii) By essentially the same methods we can also study the asymptotic behavior of solutions to (20) for more general nonlinearities. In fact, assume that h is an increasing C^1 function such that h' has only isolated erros. Moreover, assume that there are constants $\ell > 0$, $\lambda > 1$ with

$$|h(z)| < P_1(1 + |z|)^\lambda \quad , \quad H(z) > P_2|z|^{\ell+1} \quad , \quad z \in R \quad ,$$

where $H(z) = \int_0^z h(s)ds$ and P_1, P_2 positive constants. Then the assertions of proposition 11 remain true provided we replace condition (27) by $m < 2(\ell+1)/(\lambda-1)$.

REFERENCES

1. H. Brezis and A. Friedman. Nonlinear parabolic equations involving measures as initial conditions. J. Math. Pures et Appl. 62 (1983), 73-97.

2. A. Friedman. Partial differential equations. Holt, Rinehart and Winston, New York 1969.

3. J. Hale. Functional differential equations. Applied Mathematical Sciences 3, Springer-Verlag, New York 1971.

4. D. Henry. Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics 840, Springer-Verlag, Berlin 1981.

5. O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'tseva. Linear and quasilinear equations of parabolic type. Amer. Math. Soc. Transl. Math. Monographs 23, Providence, RI 1968.

6. O. A. Ladyzhenskaja and N. N. Ural'tseva. Linear and quasilinear elliptic equations. Academic Press, New York 1968.

7. P. de Mottoni, E. Orlandi and A. Tesei. Asymptotic behavior for a system describing *epidemics with migration and spatial spread of infection.* Nonl. Analysis TMA 3 (1979), 663-675.

8. R. Redheffer and W. Walter. On parabolic systems of the Volterra prey-predator type. Nonl. Analysis TMA 7 (1983), 333-347.

9. R. Redheffer, R. Redlinger and W. Walter. A theorem of LaSalle-Liapunov type for parabolic systems. Manuscript.

10. R. Redlinger. Über die C^2-Kompaktheit der Bahn von Lösungen semilinearer parabolischer Systeme. Proc. Roy. Soc. Edinburgh 93A (1982), 99-103.

11. R. Redlinger. Existence theorems for semilinear parabolic systems with functionals. Nonl. Analysis TMA 3 (1984), 667-682.

12. F. Rothe. Global solutions of reaction-diffusion systems. Lecture Notes in Mathematics 1072, Springer-Verlag, Berlin 1984.

13. P. E. Sobolevskij. Equations of parabolic type in a Banach space. Amer. Math. Soc. Transl. (2) 49 (1966), 1-62.

14. H. B. Stewart. Generation of analytic semigroups by strongly elliptic operators. TAMS 199 (1974), 141-162.

15. H. B. Stewart. Generation of analytic semigroups by strongly elliptic operators under general boundary conditions. TAMS 259 (1980), 299-310.

16. C. C. Travis and G. F. Webb. Existence and stability for partial differential equations. TAMS 200 (1974), 395-418.

17. C. C. Travis and G. F. Webb. Partial differential equations with deviating arguments in the time variable. J. Math. Anal. Appl. 56 (1976), 397-409.

RR/jvs

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2897	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* Orbital Compactness and Asymptotic Behavior of Nonlinear Parabolic Systems with Functionals		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(*s*) Reinhard Redlinger		8. CONTRACT OR GRANT NUMBER(*s*) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE January 1986
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15*a.* DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Parabolic systems with functionals, orbital compactness, asymptotic behavior

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*
  Weakly coupled semilinear parabolic systems of the form $\dot{u} - A(x)u = g(u_t)$
with homogeneous boundary conditions are studied. The nonlinear function

$g : C([-r,0] \times \bar{\Omega}, \mathbb{R}^n) \to \mathbb{R}^n$ is assumed to be locally Lipschitz continuous with

$r \geqslant 0$ a given real number and $\Omega \subset \mathbb{R}^m$ a bounded domain, $\dot{u} = du/dt$, u_t for
$t \geqslant 0$ is defined by $u_t(\sigma,\xi) = u(t+\sigma,\xi)$, $-r \leqslant \sigma \leqslant 0$, $\xi \in \bar{\Omega}$ and A is a

uniformly elliptic second order diagonal operator. Let u be a bounded
classical solution. We first establish precompactness results for the orbit

DD FORM
1 JAN 73 **1473** EDITION OF 1 NOV 65 IS OBSOLETE   UNCLASSIFIED (continued)

of u in several function spaces. Using these results and assuming that a Liapunov function V for the corresponding ordinary functional differential equation $\dot{z} = g(z_t)$ is known, we then show under some general conditions that the limit set ω^+ (as $t \to \infty$) of u consists of spatially homogeneous functions only. Moreover, ω^+ is invariant with respect to $\dot{z} = g(z_t)$ and $\dot{V} = 0$ on ω^+. The theory is illustrated with an example.

END

DTIC

6 - 86